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A NOTE ON YAMAKAWA'S QUESTION

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ABSTRACT. This short note gives an affirmative answer to a question raised by R. Yamakawa. This is included also in a result by Anisiu and Mocanu.

In the workshop on univalent functions held at RIMS during March 15-18, 1999, Professor Rikuo Yamakawa asked us the following question: Let $m_0 = 2.8329\ldots$ be the smallest number m satisfying $m > 1$ and

$$\cos(\sqrt{m^2-1}) + \sqrt{m^2-1}\sin(\sqrt{m^2-1}) = e^{-1}$$

(cf. [2]). *Is the analytic function $q(z) = m_0 z e^{m_0 z}/(e^{m_0 z} - 1)$ subordinate to the function $1 + m_0 z$ in the unit disk?* For analytic functions f and g in the unit disk we will say here that f is *subordinate* to g if there is a holomorphic map ω from the unit disk into itself such that $\omega(0) = 0$ and $f = g \circ \omega$.

The above number m_0 is nowadays known as the possible smallest one such that the condition $|f''/f'| < m_0$ for an analytic function f in the unit disk implies starlikeness of f, i.e., $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ (see [1], [3] and [4]). Yamakawa asserted that the affirmative answer to the above question can be used in an important step of a proof of the above-mentioned statement and its generalizations.

In this short note, we will provide an elementary proof for the affirmative answer to Yamakawa's question. Actually, we will prove the following result.

Proposition 1. *For $0 < m \le 3\pi/2 = 4.71\ldots$, the function $q_m(z) = mze^{mz}/(e^{mz} - 1)$ is subordinate to $1 + mz$ in the unit disk.*

Remark. The same result for $0 < m \le \pi$ was already known by Anisiu and Mocanu [1, Lemma 3]. This weaker result is sufficient to answer to Yamakawa's question. Their proof uses the Taylor expansion of $z/(e^z - 1)$ in terms of the Bernoulli numbers.

By the maximum modulus principle, we can translate the above statement as follows.

$$\begin{aligned}
&q_m(z) \text{ is subordinate to } 1 + mz \text{ in the unit disk}\\
\Leftrightarrow &\left|\frac{ze^z}{e^z-1} - 1\right| = \left|\frac{z}{1-e^{-z}} - 1\right| < m \quad \text{in } |z| < m\\
\Leftrightarrow &\left|\frac{z}{e^z-1} - 1\right| < m \quad \text{in } |z| < m\\
\Leftrightarrow &\left|\frac{1}{e^z-1} - \frac{1}{z}\right| = \left|\frac{e^z-1-z}{z(e^z-1)}\right| < 1 \quad \text{in } |z| < m\\
\Leftrightarrow &|e^z - 1 - z| \le |z||e^z - 1| \quad \text{in } |z| < m.
\end{aligned}$$

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Since the meromorphic function $F(z) := (e^z - 1 - z)/z(e^z - 1)$ is actually holomorphic in the disk $|z| < 2\pi$, it suffices to show that $|F(z)| \le 1$ in the boundary of the square $Q = \{z \in \mathbb{C}; |\mathrm{Re}\, z| < 3\pi/2, |\mathrm{Im}\, z| < 3\pi/2\}$ by the maximum modulus principle again. First we make rough estimates.

Lemma 2. *For* $|z| > 1$, *the assertion* $|F(z)| \le 1$ *holds if* $|z|/(|z| - 1) \le |e^z - 1|$.

Proof. By assumption $(1 - |z|)|e^z - 1| + |z| \le 0$, so we obtain
$$|e^z - 1 - z| \le |e^z - 1| + |z| = |z||e^z - 1| + (1 - |z|)|e^z - 1| + |z| \le |z||e^z - 1|. \qquad \square$$

Lemma 3. *If* z *with* $\mathrm{Re}\, z < -1$ *satisfies*
$$|e^z| = e^{\mathrm{Re}\, z} \le \frac{-2\mathrm{Re}\, z - 1}{4|z|^2}$$
then $|F(z)| \le 1$ *holds.*

Proof. Note that $|z| > 1$ and $|1 + z| < |z|$ by assumption. We have
$$|z| - |1 + z| = \frac{|z|^2 - |1 + z|^2}{|z| + |1 + z|} \ge \frac{-2\mathrm{Re}\, z - 1}{2|z|} \ge 2|ze^z| \ge (1 + |z|)|e^z|.$$
Therefore we obtain
$$|e^z - 1 - z| \le |e^z| + |1 + z| \le |z|(1 - |e^z|) \le |z(1 - e^z)|. \qquad \square$$

The boundary of the square Q consists of three parts: $C_1 = \{\frac{3\pi}{2} + it; |t| \le \frac{3\pi}{2}\}, C_2 = \{-\frac{3\pi}{2} + it; |t| \le \frac{3\pi}{2}\}$ and $C_3 = \{t \pm \frac{3\pi i}{2}; |t| \le \frac{3\pi}{2}\}$. Note that every point z in ∂Q satisfies $3\pi/2 \le |z| \le 3\pi/\sqrt{2}$.

On C_1 the assumption of Lemma 2 is fulfilled because $|e^z - 1| \ge e^{3\pi/2} - 1 > 110 > |z|/(|z| - 1)$ for each $z \in C_1$. For $z \in C_2$, we have
$$\frac{-2\mathrm{Re}\, z - 1}{4|z|^2} \ge \frac{3\pi - 1}{18\pi^2} > 0.047 > e^{-3\pi/2} = 0.0089\ldots$$
and hence the assumption in Lemma 3 is valid. Thus we see that $|F(z)| \le 1$ holds on C_1 and C_2. On the other hand, we have to be careful on C_3 slightly more. For $z = t \pm \frac{3}{2}\pi i \in C_3$, we consider the function
$$f(t) := |z(e^z - 1)|^2 - |e^z - 1 - z|^2 = \left(t^2 + \frac{9\pi^2}{4}\right)(e^{2t} + 1) - \left((1 + t)^2 + \left(\frac{3\pi}{2} + e^t\right)^2\right)$$
$$= \left(t^2 + \frac{9\pi^2}{4} - 1\right)e^{2t} - 3\pi e^t - (1 + 2t).$$
If $t \le -1$ we have
$$f(t) \ge \frac{9\pi^2}{4}e^{2t} - 3\pi e^t - (1 + 2t) = \left(\frac{3\pi}{2}e^t - 1\right)^2 - (2 + 2t) \ge 0.$$
Otherwise, we have
$$f(t) \ge g(t) := \left(\frac{9\pi^2}{4} - 1\right)e^{2t} - 3\pi e^t - (1 + 2t),$$

and the function g is increasing in the interval $[-1,+\infty)$ because $g''(t) \geq g''(-1) > 21$ there and $g'(-1) = 0.27\cdots > 0$. Since $g(-1) = 0.40\cdots > 0$, we have $f(t) \geq g(t) > 0$ for $t > -1$. These mean $|F| \leq 1$ on C_3, too. Now the proof is complete.

Remarks. 1. If we set $g_m(z) = (e^{mz}-1)/m$ for $m \neq 0$, then we have

$$\frac{zg_m'(z)}{g_m(z)} = \frac{mze^{mz}}{e^{mz}-1} = q_m(z) \quad \text{and} \quad 1 + \frac{zg_m''(z)}{g_m'(z)} = 1 + mz.$$

Hence the statement that q_m is subordinate to $1+mz$ means that zg_m'/g_m is subordinate to $1+zg_m''/g_m'$.

2. By numerical experiments, we can see that the best possible value for m in Proposition 1 is 4.813762.... The following figures indicate the images of the unit disk under the mappings q_m and $1+mz$ for the values $m=\pi$ and $m=4.8$ respectively.

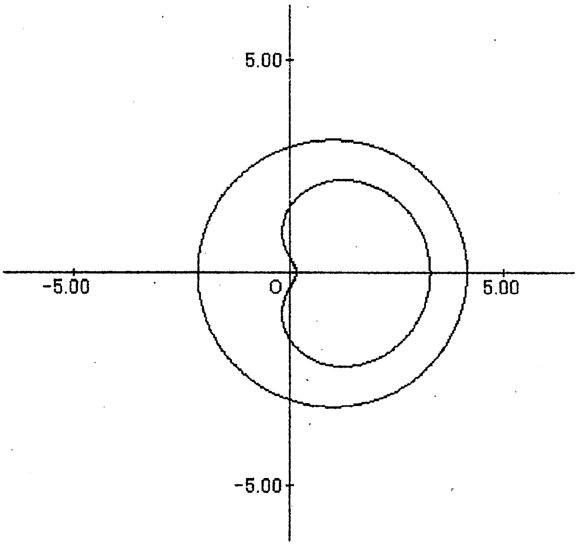

FIGURE 1. $m = 3.14$

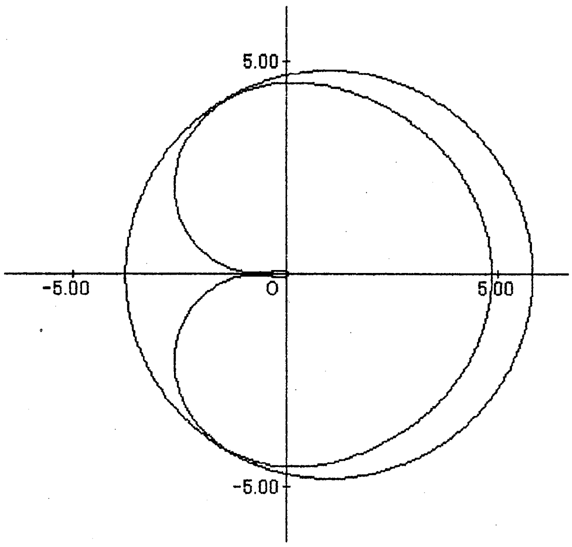

FIGURE 2. $m = 4.8$

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